

FLAT SPACETIMES

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Abstract

Those closed pseudo-Riemannian manifolds covered by Minkowski space M are classified up to finite covers. The simply transitive isometric actions on M are listed. Some spacetimes with 2 ends satisfying a causality condition are analyzed.

We call a 4-manifold X with a metric g of signature $(+, +, +, -)$ a spacetime. We will assume that g has zero curvature, i.e. X is flat. Then X is a special kind of affine manifold, namely an affine manifold with a parallel metric of the given signature. We also assume X is complete in the sense that geodesics on X extend for all time. This implies that the universal cover of X is Minkowski space M .

We will classify such X 's under the assumption that X is compact. We prove in §1 that π has a solvable subgroup of finite index, using theory developed in [5] with W. Goldman. This result has been extended to higher dimensions by Goldman and Kamashima [6] but our proof is more geometric. For a noncompact counterexample see [8] and for further discussion see [9].

The classification also uses a theorem of Auslander's on unipotent simply transitive affine actions [1]. For subgroups of the isometry group \mathcal{P} of Minkowski space, those are classified in §2. This is extended to all simply transitive actions in §3. Then in §4 we give our classification. It extends to dimension 4 that given by Auslander and Markus for 3-manifolds [2].

It is conceivable that if X is compact then g is automatically complete. A counterexample would be a very interesting spacetime: its curvature and global topology would not account for its failure to be complete. It would also be a valuable example in the theory of affine manifolds.

In §5 we discuss some two ended flat spacetimes with respect to their causal structure.

1. π is virtually solvable

A group with a solvable subgroup of finite index is called virtually solvable. We show

Theorem 1. *If $\pi \subset \mathcal{P}$ and π acts freely and properly discontinuously on M with compact quotient then π is virtually solvable.*

Proof. Taking an affine transformation to its linear part defines a natural homomorphism $\lambda: \mathcal{P} \rightarrow SO(3,1)$. We let $\Gamma = \lambda(\pi)$ and we let G be the algebraic hull of Γ . G is an algebraic subgroup of $SO(3,1)$. The identity component G_0 is of finite index in G , since G is algebraic. We will show G_0 is solvable.

As in [7] one knows that each $A \in \Gamma$ satisfies $\det(A - I) = 0$. As first noted by M. Hirsch, this reflects algebraically the fact that the nontrivial elements of π act without fixed points in M . It follows that this nontrivial polynomial equation holds on G . This shows $\dim G_0 < \dim SO(3,1)$.

Suppose G_0 is not solvable. Then it contains a semisimple connected subgroup S . As $\dim S < \dim SO(3,1)$, S is either $SO(3)$ or $SO(2,1)_0$, in properly chosen coordinates on M . In either case, S is maximal among the connected Lie subgroups of $SO(3,1)_0$, so $G_0 = S$. Thus G_0 fixes a vector v of nonzero length.

Let π_0 be the kernel of the natural map $\pi \rightarrow G/G_0$. Let $X_0 = M/\pi_0$ be the corresponding finite cover of our given spacetime X . Let \tilde{Y} be the parallel vector field on M determined by v and let Y be the corresponding vector field on X_0 . The 1-form ω on X_0 dual to Y is parallel and hence closed.

Perturb ω to a closed 1-form ω_1 with rational periods P , where P is the set of real numbers obtained by integrating ω_1 around closed loops in X_0 . As π is finitely generated, P is discrete and \mathbf{R}/P is a circle. Also $\omega_1(Y)$ never vanishes, assuming ω_1 is close enough to ω , since v has nonzero length.

Let $b \in X_0$ be a basepoint and define $\theta: X_0 \rightarrow \mathbf{R}/P$ as the indefinite integral $\theta(y) = \int_b^y \omega_1$. Then θ is a fibration of X_0 over circle [11]. Let K be a connected component of a fiber of θ . Then K is a connected cross-section to the flow φ on X_0 generated by Y .

Since Y is parallel and X_0 is flat, the flow φ has a transverse affine structure that induces an affine structure on K . Lifting φ to the universal cover M one obtains the one parameter group $\tilde{\varphi}$ of translations of M with velocity v . So \tilde{K} is naturally identified with the orbit space of this flow $M/\mathbf{R}v$, cf. [4]. This orbit space is just an affine 3-space so K is complete in its induced affine structure.

A complete compact affine 3 manifold has solvable fundamental group by [5]. Thus $\pi_1 K$ is solvable. K is the fiber of a fibration of X_0 over the circle so

the homotopy exact sequence of this fibration shows that π_0 is an extension of $\pi_1 S^1 = \mathbf{Z}$ by $\pi_1 K$. Hence π_0 is solvable. q.e.d.

It follows that the linear holonomy of X_0 , i.e. the subgroup $\Gamma_0 = \lambda(\pi_0) \subset SO(3, 1)$, is also solvable.

Now since Γ_0 has finite index in Γ , the algebraic hull $G(\Gamma_0) \subset G$ has finite index. Thus $G_0 \subset G(\Gamma_0)$, so G_0 is solvable too.

2. Unipotent groups

We will consider a subgroup $U \subset \mathcal{P}$ that is unipotent, i.e. if $Ax + v$ is an affine transformation in U then the only eigenvalue of A is 1. We suppose also that the action of U on M is simply transitive, i.e. that given $m_1, m_2 \in M$ there is exactly one $u \in U$ with $um_1 = m_2$.

Clearly such a U can be used to construct flat spacetimes $X = M/\pi$ for any discrete subgroup $\pi \subset U$. By a different procedure, these U 's arise from any compact flat spacetime. We need to classify them. We write $(A|v)$ for the infinitesimal affine motion with linear part A and translational part v .

Theorem 2. *Let U be a unipotent subgroup of \mathcal{P} that acts simply transitively on M . Then $U = \exp(L)$ where L is a nilpotent Lie algebra of infinitesimal isometries of Minkowski space M .*

In suitable linear coordinates $(w, x, y, z) = v$ on M , g is given by $g(v, v) = 2wz + x^2 + y^2$ and $L = L_{\beta, \epsilon}$ is all pairs $(A|v)$ where

$$A = \begin{pmatrix} 0 & -\beta y & -\epsilon z & 0 \\ 0 & 0 & 0 & \beta y \\ 0 & 0 & 0 & \epsilon z \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

for fixed $\beta \geq 0, \epsilon \geq 0$. The parameters β, ϵ are uniquely determined by U except for the rescaling

$$(\beta, \epsilon) \rightarrow (\lambda\beta, \lambda^2\epsilon), \quad \lambda > 0.$$

Proof. Since U is simply transitive, it is simply connected. One can identify the affine motions of 4-space with the linear motions of 5-space that preserve an affine hyperplane. Thus we may regard U as a linear unipotent group and take logarithms of the elements in U to generate L . L is a Lie algebra of nilpotent matrices that represent infinitesimal isometries of Minkowski space. Since U is locally transitive, the correspondence $(A|v) \rightarrow v$ from L to \mathbf{R}^4 is surjective. As U is four dimensional, it is a bijective correspondence and we can write $A = A(v)$. We proceed to reduce A to the indicated normal form.

We first show

Lemma 1. $\lambda(U)$ fixes a lightlike vector v_0 .

Proof. By Engels' theorem, some vector v_0 is fixed by $\lambda(U)$ (i.e., annihilated by the nilpotent Lie algebra $\lambda(L)$).

If v_0 is timelike then $\lambda(U)$ preserves v_0^\perp so $\lambda(U) \subset SO(3)$. Thus U preserves a Euclidean inner product on \mathbf{R}^4 . As a unipotent orthogonal matrix is trivial, this implies $\lambda(U) = 1$.

If v_0 is spacelike then v_0^\perp is Minkowski 3-space and we have $\lambda(U) \subset SO(2, 1)$. If $\lambda(U)$ does not fix a lightlike vector in v_0^\perp , one can again find a fixed spacelike vector v_1 in v_0^\perp . In $\{v_0, v_1\}^\perp = SO(1, 1)$ one sees that the unipotent group $\lambda(U)$ is trivial. q.e.d.

We choose linear coordinates w, x, y, z on \mathbf{R}^4 so that v_0 is the unit vector in the w direction and so that g has the form $2wz + x^2 + y^2$. We also write $\mathbf{R}^4 = \mathbf{R} + \mathbf{R}^2 + \mathbf{R}$, $v = v_1 + v_2 + v_3$ so that v_1, v_3 are 1-vectors and v_2 a 2-vector. Then if J denotes the linear map that switches v_1 and v_3 and fixes v_2 the infinitesimal isometry A for g is a solution to $A'J + JA = 0$. As A is nilpotent, it must have the block form

$$A(v) = \begin{pmatrix} 0 & -\gamma' & 0 \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

This defines in our setting a linear map $\gamma(v)$, $\gamma: \mathbf{R}^4 \rightarrow \mathbf{R}^2$ that we may express as $\gamma(v) = \gamma_1(v_1) + \gamma_2(v_2) + \gamma_3(v_3)$.

Lemma 2. $\gamma_1 = 0, \gamma_2^2 = 0$.

Proof. For any v, v' the commutator of $(A(v)|v)$ and $(A(v')|v')$ is of the form $(0|v'')$. Since L is a Lie algebra, we must have $\gamma(v'') = 0$.

Explicitly $v_1'' = -\gamma(v)v_2' + \gamma(v')v_2, v_2'' = \gamma(v)v_3' - \gamma(v')v_3$, and $v_3'' = 0$.

Let $v_3 = v_3' = v_2' = 0$. Then $0 = \gamma(v'') = \gamma_1(\gamma_1(v_1') \cdot v_2)$. Letting v_1', v_2 vary we see $\gamma_1 = 0$.

Now take $v_3 = 0$ only. Then $0 = \gamma(v'') = \gamma_2(\gamma_2(v_2)v_3')$, where we have used $\gamma_1 = 0$. Letting v_2 and v_3' vary shows $\gamma_2^2 = 0$.

Conversely if $\gamma_1 = \gamma_2^2 = 0$ then indeed $\gamma(v'') = 0$. q.e.d.

So U is determined by $\gamma_2: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ and $\gamma_3: \mathbf{R} \rightarrow \mathbf{R}^2$ with $\gamma_2^2 = 0$. We regard γ_3 as a 2-vector. One can make the following changes in γ_2, γ_3 by g -isometries that preserve the subspace $\mathbf{R}v_0$:

- 1) $\gamma_2' = \lambda\gamma_2, \gamma_3' = \lambda^2\gamma_3, \lambda \in \mathbf{R}, \lambda \neq 0$,
- 2) $\gamma_3' = B\gamma_3, \gamma_2' = B\gamma_2B^{-1}, B \in O(2)$,
- 3) $\gamma_2' = \gamma_2, \gamma_3' = \gamma_3 - \gamma_2(u), u \in \mathbf{R}^2$.

Here 1) represents a rescaling of the w and z axes, 2) represents a rigid motion of the $x - y$ plane and 3) a change of coordinates that fixes v_0^\perp (the $w - x - y$ space) and changes the z -axis.

Using 2) we can put the matrix γ_2 into the form $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$, so $\gamma_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \beta y \\ 0 \end{pmatrix}$.

Suppose $\beta \neq 0$. Then 3) can be used to put γ_3 into the form $\begin{pmatrix} 0 \\ \epsilon \end{pmatrix}$. If $\beta = 0$ then a) can be used to the same effect. So we reach the normal form for γ and A . It is not hard to arrange $\beta, \epsilon \geq 0$ by using $\lambda = -1$ in 1) and $B = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ in 2) if necessary.

We now show that (β, ϵ) and (β', ϵ') give the same U only if $\beta' = \lambda\beta, \epsilon' = \lambda^2\epsilon$, for some $\lambda > 0$. The case $\beta = \epsilon = 0$ is uniquely characterized by the property $\lambda(U) = 1$.

Lemma 3. *If $(\beta, \epsilon) \neq (0, 0)$ then the w -axis W consists of all the lightlike vectors fixed by $\gamma(U)$.*

Proof. Suppose $v = (w_1, x_1, y_1, z_1)$ is lightlike and fixed, i.e. $2w_1z_1 + x_1^2 + y_1^2 = 0, -\beta yx_1 + \epsilon zy_1 = 0, \beta yz_1 = 0, \text{ and } \epsilon zz_1 = 0$ for all $y, z \in \mathbf{R}$. Whether $\beta \neq 0$ or $\epsilon \neq 0$, one has $z_1 = 0$. This gives $x_1^2 + y_1^2 = 0$ so $x_1 = y_1 = 0$. $\therefore v = (w_1, 0, 0, 0)$. q.e.d.

Thus W and its orthogonal the $w, x, y, 3$ -space W^\perp are determined by U when $\gamma(U) \neq 1$. It is now easy to check that only the coordinate changes 1), 2) and 3) above are relevant for comparing (β, ϵ) and (β', ϵ') . The special case $\beta = 0$ corresponds to $\gamma|_{W^\perp} = 0$. The special case $\epsilon = 0$ corresponds to the other possibility for $\text{rank}(\gamma: \mathbf{R}^4 \rightarrow \mathbf{R}^2) < 2$. It only remains to analyze the cases $\beta > 0, \epsilon > 0$: but one can check that choosing $\epsilon = 1$ forces the value of β . q.e.d.

If $(\beta, \epsilon) \neq (0, 0)$, we call the corresponding unipotent group U_ρ where $\rho = \beta^2/\epsilon \in [0, \infty]$. Thus the only U 's are the translation group T and the U_ρ 's.

3. Simply transitive actions

We will consider a Lie group $H \subset \mathcal{P}$ that acts simply transitively on M . Then H must be solvable $[A, M]$. Auslander proved that there is an associated unipotent group U that also acts simply transitively on M , namely the unipotent radical of the algebraic hull G of $H[A]$. We will use this fact, together with the results of the previous section, to find all H 's.

Suppose U is not the translation group T , so U corresponds to a nonzero pair (β, ϵ) . If $\beta = 0$ there is a two dimensional space of parallel vector fields corresponding to the $w - x$ plane. If $\beta \neq 0$ then the $w - x$ plane is determined as the kernel of γ . Thus the flag $W \subset (w - x \text{ plane}) \subset W^\perp$ is in

both cases determined by U and the normalizer N of U preserves this flag. In particular elements of N have only real eigenvalues.

Since U is the unipotent radical of G , $U \triangleleft G$. Thus $G \subset N$. Let L_G be the Lie algebra of G . Then elements of L_G have the form $(A|v)$ where

$$A = \begin{pmatrix} \lambda & -\delta^* & 0 \\ 0 & 0 & \delta \\ 0 & 0 & -\lambda \end{pmatrix}$$

relative to the splitting $\mathbf{R}^4 = \mathbf{R} \oplus \mathbf{R}^2 \oplus \mathbf{R}$. The middle block vanishes because G preserves the flag discussed above.

We will show $\lambda = 0$. Subtracting off an element of $L_{\beta, \epsilon}$ we may suppose $v = 0$. Rescaling, we may suppose $\lambda = 1$. By choosing the z axis differently we may suppose $\delta = 0$. Computing the commutator of $(A|0)$ with the typical element $(A(v)|v) \in L_{\beta, \epsilon}$ gives (in this new basis which is like that used in Lemma 2)

$$\left[\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right), \left(\begin{array}{ccc|c} 0 & -\gamma' & 0 & v_1 \\ 0 & 0 & \gamma & v_2 \\ 0 & 0 & 0 & v_3 \end{array} \right) \right] = \left(\begin{array}{ccc|c} 0 & -\gamma' & 0 & v_1 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & -v_3 \end{array} \right).$$

Since $(A|0)$ must normalize $L_{\beta, \epsilon}$, we must have $\gamma(v_1, 0, -v_3) = \gamma(v_1, v_2, v_3)$. This gives $\gamma_2 = \gamma_3 = 0$, so $\beta = \epsilon = 0$ contrary to assumption. So $\lambda = 0$.

It follows that L_G is nilpotent, so G and H are unipotent. But any unipotent connected Lie group is Zariski closed, so $H = G$. Taking this unipotent radical we see $H = U = U_\rho$. We have

Theorem 3. *If $H \subset \mathcal{P}$ acts simply transitively on Minkowski space M then either*

- 1) H is one of the unipotent groups U_ρ , $\rho \in [0, \infty]$, or
- 2) the unipotent radical U of the algebraic hull G of H is precisely the group T of translations of M .

We now restrict to case 2) with $H \neq T$, i.e. we suppose H is not unipotent. On the one hand $\lambda(H) = H/H \cap T \subset G/T = G/U$. The quotient G_0/U is abelian (indeed it is isomorphic to a linear group of diagonal matrices). Thus $\lambda(H)$ is abelian. A nontrivial connected abelian subgroup of $SO(3,1)$ has dimension $d \leq 2$. As the translation subgroup $H \cap T$ is normalized by H , $\lambda(H)$ preserves the corresponding subspace of M . We find

Theorem 4. *If $H \subset \mathcal{P}$ acts simply transitively on M then either H is unipotent (and so described by Theorem 2) or $\lambda(H)$ is a nonunipotent 1-parameter subgroup of $\lambda(\mathcal{P}) = SO(3,1)$ and $H \cap T = \ker(\lambda|H)$ is a 3-*

dimensional invariant subspace of $\lambda(H)$. In the latter case there are linear coordinates w, x, y, z on M for which $g = 2wz + x^2 + y^2$ and the Lie algebra L_H consists of all pairs $(A|v)$ where

$$\text{a) } v = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \quad A(v) = \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \end{pmatrix},$$

or

$$\text{b) } v = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}, \quad A(v) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Suppose $H \cap T$ has dimension 3 and choose $A_0 \neq 0$ in the Lie algebra of $\lambda(H)$. A_0 is not nilpotent.

As in the proof of Theorem 1, $\det(\Lambda - I) = 0$ for all $\Lambda \in \lambda(H)$. Thus $\det A_0 = 0$, that is A_0 is singular.

As A_0 is an infinitesimal isometry, its eigenvalues occur in pairs $\pm\mu$. We must have exactly two eigenvalues equal to 0. Rescaling A_0 , we may suppose the other two are $\pm i$ or ± 1 .

If they are $\pm i$, the eigenspace they span is irreducible. The invariant subspace $H \cap T$ has codimension one and so must contain this eigenspace. This gives case b), where w, x, y span $H \cap T$ and the $\pm i$ eigenspace is spanned by x, y .

If the eigenvalues are ± 1 and $H \cap T$ contains both eigenspaces, one gets case a). Otherwise, switching to $-A_0$ if necessary, we may suppose $H \cap T$ contains the $+1$ and 0 eigenspaces. This leads to a nontransitive affine group, so it does not contribute to our list.

If $H \cap T$ has dimension 2, then the Lie algebra of $\lambda(H)$ consists of singular commuting elements. This easily implies that $\lambda(H)$ is unipotent. q.e.d.

We summarize our results in the following table, listing (up to conjugacy in \mathcal{P}) all the simply transitive isometric actions on Minkowski space with respect to coordinates in which $g = 2wz + x^2 + y^2$. The only redundancy is that $(\beta, \epsilon) \neq (0, 0)$ and $(\beta', \epsilon') \neq (0, 0)$ determine the same U_ρ if $\rho = \beta^2/\epsilon = (\beta')^2/\epsilon'$. We show a typical element of each group where $r, s, t, u \in \mathbf{R}$. $(A|v)$ denotes the motion $p \rightarrow Ap + v, p \in \mathbf{R}^4$.

Simply Transitive Motions of Minkowski Space

$$T: \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & r \\ 0 & 1 & 0 & 0 & s \\ 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 1 & u \end{array} \right).$$

U_ρ :

$$\left(\begin{array}{cccc|c} 1 & -\beta t & -\epsilon u & -\frac{1}{2}(\beta^2 t^2 + \epsilon^2 u^2) & r - \frac{1}{2}(\beta s t + \epsilon t u) - \frac{1}{2}(\beta^2 t^2 u + \epsilon^2 u^3) \\ 0 & 1 & 0 & \beta t & s + \frac{1}{2}\beta t u \\ 0 & 0 & 1 & \epsilon u & t + \frac{1}{2}\epsilon u^2 \\ 0 & 0 & 0 & 1 & u \end{array} \right).$$

$$\mathcal{A}: \left(\begin{array}{cccc|c} e^t & 0 & 0 & 0 & r \\ 0 & 1 & 0 & 0 & s \\ 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & e^{-t} & u \end{array} \right).$$

$$\mathcal{B}: \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & r \\ 0 & \cos u & \sin u & 0 & s \\ 0 & -\sin u & \cos u & 0 & t \\ 0 & 0 & 0 & 1 & u \end{array} \right).$$

4. Discrete groups

Let π be a discrete subgroup of a simply transitive group H of motions of Minkowski space M . Then the orbit space $X = M/\pi$ is a flat spacetime. The possible discrete subgroups π depend only on the structure of H as a Lie group and not on its embedding in \mathcal{P} . Since H is known up to conjugacy by §3, we can easily find the possible subgroups π .

By a theorem in [5], every compact complete affine manifold $X = A/\Gamma$ with virtually solvable Γ has a *crystallographic hull* $H(\Gamma)$. This group $H(\Gamma)$ has an identity component $H_0(\Gamma)$ that acts simply transitively on A . Also $H(\Gamma)$ has only finitely many components, each of which meets Γ . These properties determine $H(\Gamma)$ uniquely in case Γ (or some subgroup of finite index in Γ) consists of matrices with only real eigenvalues.

We turn to the special case $A = M$, $\Gamma \subset \mathcal{P}$. Then A/Γ is the most general compact complete flat spacetime, since §1 assures us that Γ is virtually solvable. The identity component $H_0(\Gamma)$ must occur in our table of simply transitive motions. By Bieberbach's theorem, any discrete subgroup of B meets

T in a subgroup of finite index [12]. Thus in all cases Γ has a subgroup of finite index with only real eigenvalues. So the hull $H(\Gamma)$ is unique in our setting. If $\pi = \Gamma \cap H_0(\Gamma)$ and $H = H_0(\Gamma)$, the quotient M/π is a finite regular cover of A/Γ , canonically associated to Γ . So by classifying the discrete subgroups π , we will classify up to a natural finite cover all the compact, complete, flat spacetimes.

Now we lose little by not discussing \mathcal{P} further, since it does not arise as $H_0(\Gamma)$. The case $H = T$ is obvious: π is any lattice in \mathbf{R}^4 . Only the cases $H = U_\rho$ and $H = \mathcal{A}$ give interesting groups.

Theorem 5. *X is a flat complete compact spacetime with fundamental group Γ . There is a uniquely determined simply transitive group H of motions of the universal cover \tilde{X} such that $H \cap \Gamma = \pi$ has finite index in Γ .*

The group π is a semidirect product $\mathbf{Z}^3 \times_A \mathbf{Z}$. $A \in SL(3, \mathbf{Z})$ has characteristic polynomial $p_A(t) = \det(t - A) = (t - 1)(t^2 - bt + 1)$, with $b \geq 2$ an integer.

The similarity class of A over \mathbf{Q} (or \mathbf{C}) determines the group H up to isomorphism as a Lie group. In the finer classification as a subgroup of \mathcal{P} , H is conjugate to

- a) \mathcal{A} , if $b > 2$,
- b) U_ρ with $\rho > 0$, if $(A - I)^3 = 0, (A - I)^2 \neq 0$,
- c) U_0 , if $(A - I)^2 = 0, A - I \neq 0$,
- d) T , if $A = I$.

In each case, if $\alpha = \log A$ then H is the semidirect product of \mathbf{R}^3 by \mathbf{R} where \mathbf{R} acts on \mathbf{R}^3 by $e^{t\alpha}$, $t \in \mathbf{R}$.

Proof. The first paragraph has been shown and all possibilities for H are known.

In each case H has an abelian subgroup H' isomorphic to \mathbf{R}^3 . For \mathcal{A} there is the subgroup $T \cap \mathcal{A}$, for U_ρ the subgroup defined by $t = 0$. This shows that H is a semidirect product by some one parameter group $e^{t\alpha}$. One sees that α has eigenvalues all 0 if $H = U_\rho$ or $H = T$ and eigenvalues $0, \pm\lambda$, some $\lambda \in \mathbf{R}$, if $H = \mathcal{A}$.

In case $H = \mathcal{A}$, one can check that $H' = T \cap \mathcal{A}$ is the maximum connected normal nilpotent subgroup of H . By [10, 3.5] π meets H' in a lattice. Since $H' \cong \mathbf{R}^3$, $\pi \cap H' \cong \mathbf{Z}^3$. The quotient $\pi/\pi \cap H'$ is discrete in $H/H' \cong \mathbf{R}$, so it is infinite cyclic. The action A of \mathbf{Z} on \mathbf{Z}^3 has eigenvalues $\mu, 1, \mu^{-1}$, $\mu > 1$ and so $\det(t - A)$ is of form $(t - 1)(t^2 - bt + 1)$, $b \geq 3$.

In the unipotent cases, the argument is similar. One only needs to show that H' can be chosen so $\pi \cap H'$ is a lattice in H' .

For $H = T$, this is trivial.

For $H = U_0$ (i.e. $\beta = 0, \varepsilon = 1$) the center Z of H is 2-dimensional. By [10, 2.17] $Z \cap \pi$ is a lattice in Z . The quotient $\pi/Z \cap \pi$ is a lattice in \mathbf{R}^2 . Taking an element of π that corresponds to an indivisible vector in $\pi/Z \cap \pi$ and adjoining it to $Z \cap \pi$ gives a free abelian subgroup π' of rank 3. The quotient π/π' is infinite cyclic. One takes H' to be the Malcev completion of π' .

Now suppose $H = U_\rho, \rho > 0$, say $\beta = 1$. Now the center Z is 1-dimensional and the quotient H/Z is the Heisenberg group. Let G be the discrete subgroup $\pi/\pi \cap \mathbf{Z}$ of H/Z . It is a lattice and so it has the form $\langle x, y, z \mid x \leftrightarrow y, z, yz = x^a zy \rangle$. The central extension π of G by \mathbf{Z} has presentation $\langle w, x, y, z \mid w \leftrightarrow x, y, z, xy = w^b yx, xz = w^c zx, yz = w^d x^a zy \rangle$ for suitable integers a, b, c, d . One finds immediately that $xy^m z^n = w^{bm+cn} y^m z^n x$. Choose m, n relatively prime with $bm + cn = 0$. Then the subgroup π' generated by w, x , and $y^m z^n$ is free abelian with quotient group \mathbf{Z} . Again we take H' to be the Malcev completion of π' in H . q.e.d.

The quotient groups Γ/π are not very large if $H \neq T$. For instance, recall that the groups U_ρ have a natural flag: this must be preserved by Γ . The graded action of Γ is a sum of 4×1 matrices and it has finite order and determinant 1. Thus $[\Gamma : \pi]$ divides 8. If $H = T, \Gamma$ is on the list of crystallographic groups in dimension 4, [3]. Thus the finite extensions are, in principle, routinely computable.

Note that we have shown

Corollary. *Every complete compact flat spacetime is finitely covered by a T^3 bundle over the circle.*

5. Causality

Let us call a motion $Ap + v$ of Minkowski space *causal* if the displacement vector $Ap - p + v$ is spacelike or zero for all points p . If $\pi \subset \mathcal{P}$ acts freely and properly discontinuously on M and each element of π is causal, we will say that π and $X = M/\pi$ are a causal group and a causal spacetime, respectively. Here no timelike or lightlike geodesic in X returns to its starting point, reflecting the physical notion of causality.

If π is causal, one cannot have X compact. The strongest compactness property one can assume corresponds to the compactness of the "special directions" in X . This is to assume that X has two ends, i.e. that there is a compact set C (separating X into two unbounded open components but no compact C' gives more than two. Clearly this holds if $X = K \times \mathbf{R}, K$ compact: the 2 ends correspond to those of \mathbf{R} , i.e., to $\pm\infty$, or the infinite future and infinite past.

We will further suppose that π is contained in a simply transitive group of motions H . Then we find

Theorem 6. *Let H be a simply transitive group of motions of M and let π be a causal subgroup of H such that $X = M/\pi$ has two ends. Then there is an affine fibration of X over \mathbf{R} with 3-tori as fibers. In particular π is free abelian of rank 3 and X is diffeomorphic to $T^3 \times \mathbf{R}$.*

Proof. Suppose to begin that $H = \mathcal{A}$. The element $Ap + v$ of \mathcal{A} is causal if $2(r + (e^t - 1)w)(u + (e^{-t} - 1)z) + s^2 + t^2 > 0$ for all $w, z \in \mathbf{R}$. If $t \neq 0$ this is absurd. So $t = 0$ on π . Thus $\pi \subset T$.

Similarly if $H = \mathcal{B}$, one can show $\pi \subset T$.

But for $\pi \subset T$, the condition that X has two ends means just that π has rank 3. Let T' be the vector space spanned by π . The projection $X \rightarrow M/T'$ is the desired affine fibration. The fibers are flat Euclidean tori, except in the case when $T' = v^\perp$, v lightlike, $\mathbf{R}v \cap \pi = 0$.

We now suppose $H = U_\rho$. In the notation of our table, let $(A|v) \in U_\rho$ correspond to the parameters r, s, t, u . Let p be the vector with coordinates w, x, y, z and write v_i for the coordinates of v , δ_i for those of $Ap - p + v$. Then

$$\begin{aligned} \delta_1 &= -\beta tx - \varepsilon vy - \frac{1}{2}(\beta^2 t^2 + \varepsilon^2 u^2)z + v_1, \\ \delta_2 &= \beta tz + v_2, \delta_3 = \varepsilon uz + v_3, \delta_4 = v_4 = u, \end{aligned}$$

and causality means $2\delta_1\delta_4 + \delta_2^2 + \delta_3^2 > 0$ for all x, y, z . Considering the terms in x and y , one sees that $\beta tu = \varepsilon u^2 = 0$ at all elements of π .

Suppose there is some element of π with nonzero u . Then $\varepsilon = 0$. As $(\beta, \varepsilon) \neq (0, 0)$, we have $\beta \neq 0$. Thus $tu = 0$ on π . Since $\varepsilon = 0$, the map $\phi: U_\rho \rightarrow \mathbf{R}^2$, $\phi(A|v) = \begin{pmatrix} t \\ u \end{pmatrix}$ is a group homomorphism. As $\phi(\pi)$ is a subgroup of \mathbf{R}^2 and lies in the coordinate axes, we must have $t = 0$ on π . Thus $2\delta_1\delta_4 + \delta_2^2 + \delta_3^2 = 2ru + s^2$ and $\pi \subset T$. So again π has rank 3. It follows that r, s, u can be chosen on π with $2ru + s^2 < 0$ since any cocompact lattice in Minkowski 3-space meets the interior of the light cone. This contradiction shows $u = 0$ on π . Causality holds provided s, t does not vanish simultaneously on nonzero elements of π .

Since $u = 0$, π preserves the z -coordinate. This gives the fibration of X over \mathbf{R} . One can check that $\ker u \subset U_\rho$ is an abelian subgroup, so the fiber is a 3-torus. q.e.d.

For each fixed z , the action of $\pi \subset U_\rho$ is given by

$$\left(\begin{array}{ccc|c} 1 & -t & 0 & r - \frac{1}{2}st - \frac{1}{2}t^2z \\ 0 & 1 & 0 & s + tz \\ 0 & 0 & 1 & t \end{array} \right),$$

where we have set $\beta = 1$ for simplicity (if $\beta = 0$ then $\pi \subset T$). Thus the affine structures on the fibers vary with z . They are never Euclidean. The vector field $\partial/\partial w$ gives bounded light rays on each toral fiber, so the causality of these examples is very delicate.

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